Invariants of triangular Lie algebras with one nil-independent diagonal element

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Abstract

The invariants of solvable triangular Lie algebras with one nil-independent diagonal element are studied exhaustively. Bases of the invariant sets of all such algebras are constructed using an original algebraic algorithm based on Cartan's method of moving frames and the special technique developed for triangular and closed algebras in Boyko *et al* (*J. Phys. A: Math. Theor.* 2007 **40** 7557). The conjecture of Tremblay and Winternitz (*J. Phys. A: Math. Gen.* 2001 **34** 9085) on the number and form of elements in the bases is completed and proved.

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1. Introduction

The possibility of finding complete explicit formulae for the invariants of a Lie algebra is without a doubt connected with some precise knowledge of its structure. Since the invariants of Lie algebras are their essential characteristics, and are important in their application, the exhaustive description of invariants was attempted for all known structures of Lie algebras.

This problem was solved in the cases of the semi-simple and low-dimensional Lie algebras, for physically relevant Lie algebras of fixed dimensions, as well as Lie algebras with the simplest (Abelian) radicals (see, e.g., [2, 6, 15] and references therein). Further progress in the study of Lie algebra invariants (also called generalized Casimir operators) is closely related to the progress in the classification of classes of solvable algebras and unsolvable Lie algebras with non-trivial radicals of arbitrary finite dimensions [1, 5, 6, 10–12, 17–20]. The

infinitesimal method became the convention for the computation of invariants. It is based on the integration of a linear system of first-order partial differential equations associated with infinitesimal operators of co-adjoint action. Algebraic tools were occasionally applied in the construction of invariants for special classes of algebras [9, 16].

In [2, 3], an original pure algebraic approach to invariants was proposed and developed. It involves Fels–Olver's approach to Cartan's method of moving frames [7, 8]. (For modern development of the moving frames method and more references see also [14].) More precisely, the technique of the moving frames method is specialized in its framework for the case of co-adjoint action of the associated inner automorphism groups on the dual spaces of Lie algebras. Unlike the infinitesimal method, such an approach allows us to avoid solving systems of differential equations, replacing them by algebraic equations. As a result, it is essentially simpler to apply.

Different versions of the algebraic approach were tested in [2, 3] for the Lie algebras of dimensions not greater than 6 and a wide range of known solvable Lie algebras of arbitrary finite dimensions with a fixed structure of nil-radicals. A special technique for working with solvable Lie algebras having triangular nil-radicals was developed in [4]. Fundamental invariants were constructed with this technique for the algebras $\mathfrak{t}_0(n)$, $\mathfrak{t}(n)$ and $\mathfrak{st}(n)$. Here $\mathfrak{t}_0(n)$ denotes the nilpotent Lie algebra of strictly upper triangular $n \times n$ matrices over the field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$. The solvable Lie algebras of non-strictly upper triangular and special upper triangular $n \times n$ matrices are denoted by $\mathfrak{t}(n)$ and $\mathfrak{st}(n)$ respectively.

The invariants of triangular algebras were first considered in [20], with the infinitesimal method. Theorem 1 on the Casimir operators of $\mathfrak{t}_0(n)$ and proposition 1 on the invariants of $\mathfrak{st}(n)$ from [20] were completely corroborated in [4]. Note that proposition 1 was only a conjecture derived after the calculation of the invariants for all partial values $n \leqslant 13$. Another conjecture was formulated in [20] as proposition 2 on invariant bases of solvable Lie algebras having $\mathfrak{t}_0(n)$ as their nil-radicals and possessing a minimal (one) number of nil-independent 'diagonal' elements. It was invented after the construction of the invariants for a narrower range of n than in the case of $\mathfrak{st}(n)$ (namely, $n \leqslant 8$), and it has not been proved as of this writing. In the framework of the infinitesimal approach, the necessary calculations are too cumbersome, even more so for these algebras. This probably led to the reduction of possibility of computational experiments and to the impossibility of proving the aforementioned conjectures for arbitrary values of n.

In this paper we rigorously construct bases of the invariant sets for all the solvable Lie algebras with nil-radicals isomorphic to $\mathfrak{t}_0(n)$ and one nil-independent 'diagonal' element for arbitrary relevant values of n (i.e., $n > 1$). We use the algebraic approach along with some additional technical tools that were developed for triangular and closed algebras in [4]. All the steps of the algorithm are implemented one after another: construction of the co-adjoint representation of the corresponding Lie group and its fundamental lifted invariant (section 2), excluding the group parameters from the lifted invariants by the normalization procedure that results in a basis of the invariants for the co-adjoint action (section 3) and re-writing this basis as a basis of the invariants of the Lie algebra under consideration (section 4). Description of some necessary notions and statements, precise formulation and discussion of the technical details of the applied algorithm can be found in [2–4], and hence are omitted here. The calculations involved in any step are more complicated than in [4], but due to optimization they remain quite useful. There are two cases, depending on the parameters of the algebra, that differ in the necessary number of normalization constraints and, therefore, in the cardinality of the fundamental invariants. The conjecture given in proposition 2 of [20] is completed and proved.

2. Representation of the co-adjoint action

Consider the solvable Lie algebra $\mathfrak{t}_\gamma(n)$ with the nil-radical $NR(\mathfrak{t}_\gamma(n))$ isomorphic to $\mathfrak{t}_0(n)$ and one nil-independent element f, which acts on elements of the nil-radical in the same way the diagonal matrix $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ acts on strictly triangular matrices, also consider Γ as being a matrix non-proportional to the unity matrix. The tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$ has different elements due to the condition on Γ. It is defined up to a nonzero multiplier and homogeneous shift. In other words, the algebras $\mathfrak{t}_\gamma(n)$ and $\mathfrak{t}_{\gamma'}(n)$ are isomorphic if and only if there exists $\lambda, \mu \in \mathbb{F}, \lambda \neq 0$ such that $\gamma_i' = \lambda\gamma_i + \mu, i = 1, \ldots, n$. The tuples γ and γ' are assumed to be equivalent. Up to the equivalence the additional condition $\mathrm{Tr}\,\Gamma = \sum_i \gamma_i = 0$ can be imposed on the algebra parameters. Therefore, the algebra $\mathfrak{t}_\gamma(n)$ is naturally embedded into $\mathfrak{st}(n)$ as an ideal, thus identifying $NR(\mathfrak{t}_\gamma(n))$ with $\mathfrak{t}_0(n)$ and f with Γ.

The union of the canonical basis of $NR(\mathfrak{t}_\gamma(n))$ and the one-element set $\{f\}$ is chosen as the canonical basis of $\mathfrak{t}_\gamma(n)$. In the basis of $NR(\mathfrak{t}_\gamma(n))$, we use a 'matrix' enumeration of the basis elements $e_{ij}, i < j$, with an 'increasing' pair of indices, similarly to the canonical basis $\left\{E_{ij}^n, i < j\right\}$ of the isomorphic matrix algebra $\mathfrak{t}_0(n)$.

Hereafter E_{ij}^n (for the fixed values i and j) denotes the $n \times n$ matrix $(\delta_{ii'}\delta_{jj'})$ with i' and j' running the numbers of rows and column respectively, i.e., the $n \times n$ matrix with the unit on the cross of the ith row and the jth column and zero otherwise. The indices i, j, k and l run at most from 1 to n. Only additional constraints on the indices are indicated.

Thus, the basis elements $e_{ij} \sim E_{ij}^n, i < j$ and $f \sim \sum_i \gamma_i E_{ii}^n$ satisfy the commutation relations

$$[e_{ij}, e_{i'j'}] = \delta_{i'j}e_{ij'} - \delta_{ij'}e_{i'j}, \qquad [f, e_{ij}] = (\gamma_i - \gamma_j)e_{ij},$$

where δ_{ij} is the Kronecker delta.

The Lie algebra $\mathfrak{t}_\gamma(n)$ can be considered as the Lie algebra of the Lie subgroup

$$T_\gamma(n) = \{B \in T(n) \mid \exists \varepsilon \in \mathbb{F}: b_{ii} = e^{\gamma_i \varepsilon}\}$$

of the Lie group $T(n)$ of non-singular upper triangular $n \times n$ matrices.

Let e_{ji}^*, x_{ji} and y_{ij} denote the basis element and the coordinate function in the dual space $\mathfrak{t}_\gamma^*(n)$ and the coordinate function in $\mathfrak{t}_\gamma(n)$, which correspond to the basis element $e_{ij}, i < j$. In particular, $\langle e_{j'i'}^*, e_{ij}\rangle = \delta_{ii'}\delta_{jj'}$. The reverse order of subscripts of the dual elements and coordinates is justified by the simplification of a matrix representation of lifted invariants. f^*, x_0 and y_0 denote similar objects corresponding to the basis element f. We additionally put $y_{ii} = \gamma_i y_0$ and then complete the sets of x_{ji} and y_{ij} with zeros to the matrices X and Y. Hence X is a strictly lower triangular matrix and Y is a non-strictly upper triangular one. The analogous 'matrix' with the significant elements $e_{ij}, i < j$, is denoted by $\mathcal{E}$.

Lemma 1. *A complete set of functionally independent lifted invariants of* $\mathrm{Ad}_{T_\gamma(n)}^*$ *is exhausted by the expressions*

$$\mathcal{I}_{ij} = \sum_{i \leqslant i' < j' \leqslant j} b_{ii'}\widehat{b}_{j'j}x_{i'j'}, \qquad j < i, \qquad \mathcal{I}_0 = x_0 + \sum_{j < i}\sum_{j \leqslant l \leqslant i} \gamma_l b_{li}\widehat{b}_{jl}x_{ij},$$

where $B = (b_{ij})$ is an arbitrary matrix from $T_\gamma(n)$ and $B^{-1} = (\widehat{b}_{ij})$ is the inverse matrix of B.

Proof. The adjoint action of $B \in T_\gamma(n)$ on the matrix Y is $\mathrm{Ad}_B Y = BYB^{-1}$, i.e.,

$$\mathrm{Ad}_B\left(y_0 f + \sum_{i<j} y_{ij}e_{ij}\right) = y_0 f + y_0 \sum_{i<j}\sum_{i \leqslant i' \leqslant j} b_{ii'}\gamma_i\widehat{b}_{i'j}e_{ij} + \sum_{i \leqslant i' < j' \leqslant j} b_{ii'}y_{i'j'}\widehat{b}_{j'j}e_{ij}.$$

After changing $e_{ij} \to x_{ji}$, $y_{ij} \to e_{ji}^*$, $f \to x_0$, $y_0 \to f^*$, $b_{ij} \leftrightarrow \widehat{b}_{ij}$ in the latter equality, we obtain the representation for the co-adjoint action of B:

$$\mathrm{Ad}_B^*\left(x_0 f^* + \sum_{i<j} x_{ji} e_{ji}^*\right) = x_0 f^* + \sum_{i<j} \sum_{i \leqslant i' \leqslant j} b_{i'j} x_{ji} \widehat{b}_{ii'} \gamma_{i'} f^* + \sum_{i \leqslant i' < j' \leqslant j} b_{j'j} x_{ji} \widehat{b}_{ii'} e_{j'i'}^*$$

$$= \left(x_0 + \sum_{i<j} \sum_{i \leqslant i' \leqslant j} b_{i'j} x_{ji} \widehat{b}_{ii'} \gamma_{i'}\right) f^* + \sum_{i'<j'} (BXB^{-1})_{j'i'} e_{j'i'}^*.$$

Therefore, $\mathcal{I}_0$ and the elements $\mathcal{I}_{ij}$, $j < i$, of the matrix $\mathcal{I} = BXB^{-1}$, where $B \in T_\gamma(n)$, form a fundamental lifted invariant of $\mathrm{Ad}_{T_\gamma(n)}^*$. $\qquad\square$

Note 1. The complete set of parameters in the above representation of lifted invariants is formed by b_{ij}, $j < i$, and ε. The center of the group $T_\gamma(n)$ is nontrivial only if $\gamma_1 = \gamma_n$, namely, then $Z(T_\gamma(n)) = \left\{E^n + b_{1n} E_{1n}^n, b_{1n} \in \mathbb{F}\right\}$. Here $E^n = \mathrm{diag}(1, \ldots, 1)$ is the $n \times n$ unity matrix. In this case the inner automorphism group of $\mathfrak{t}_\gamma(n)$ is isomorphic to the factor group $T_\gamma(n)/Z(T_\gamma(n))$ and hence its dimension is $\frac{1}{2}n(n-1)$. The parameter b_{1n} in the representation of the lifted invariants is thus inessential. Otherwise, the inner automorphism group of $\mathfrak{t}_\gamma(n)$ is isomorphic to the whole group $T_\gamma(n)$, and all the parameters in the constructed lifted invariants are essential.

3. Invariants of the co-adjoint action

Below $A_{j_1,j_2}^{i_1,i_2}$, where $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, denotes the submatrix $(a_{ij})_{j=j_1,\ldots,j_2}^{i=i_1,\ldots,i_2}$ of a matrix $A = (a_{ij})$. The standard notation $|A| = \det A$ is used. The conjugate value of k with respect to n is denoted by $\varkappa$, i.e., $\varkappa = n - k + 1$.

At first we formulate the technical lemma from [4], applied to the proof of the following theorem.

Lemma 2. *Suppose* $1 < k < n$. *If* $\left|X_{1,k-1}^{\varkappa+1,n}\right| \neq 0$ *then for any* $\beta \in \mathbb{F}$

$$\left(\beta - X_{1,k-1}^{i,i}\left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{j,j}^{\varkappa+1,n}\right) = \frac{(-1)^{k+1}}{\left|X_{1,k-1}^{\varkappa+1,n}\right|} \begin{vmatrix} X_{1,k-1}^{i,i} & \beta \\ X_{1,k-1}^{\varkappa+1,n} & X_{j,j}^{\varkappa+1,n} \end{vmatrix}.$$

In particular, $\left(x_{\varkappa k} - X_{1,k-1}^{\varkappa,\varkappa}\left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{k,k}^{\varkappa+1,n}\right) = (-1)^{k+1}\left|X_{1,k-1}^{\varkappa+1,n}\right|^{-1}\left|X_{1,k}^{\varkappa,n}\right|$. *Analogously*

$$\left(x_{\varkappa j} - X_{1,k-1}^{\varkappa,\varkappa}\left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{j,j}^{\varkappa+1,n}\right)\left(x_{jk} - X_{1,k-1}^{j,j}\left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{k,k}^{\varkappa+1,n}\right)$$

$$= \frac{1}{\left|X_{1,k-1}^{\varkappa+1,n}\right|} \begin{vmatrix} X_{1,k}^{j,j} & \beta \\ X_{1,k}^{\varkappa,n} & X_{j,j}^{\varkappa,n} \end{vmatrix} + \frac{\left|X_{1,k}^{\varkappa,n}\right|}{\left|X_{1,k-1}^{\varkappa+1,n}\right|^2} \begin{vmatrix} X_{1,k-1}^{j,j} & \beta \\ X_{1,k-1}^{\varkappa+1,n} & X_{j,j}^{\varkappa+1,n} \end{vmatrix}.$$

Theorem 1. *A basis of* $\mathrm{Inv}\left(\mathrm{Ad}_{T_\gamma(n)}^*\right)$ *is formed by the expressions*

(1) $\left|X_{1,k}^{\varkappa,n}\right|, k = 1, \ldots, \left[\frac{n}{2}\right], \qquad x_0 + \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(-1)^{k+1}}{\left|X_{1,k}^{\varkappa,n}\right|}(\gamma_k - \gamma_{k+1}) \sum_{k<i<\varkappa} \begin{vmatrix} X_{1,k}^{i,i} & 0 \\ X_{1,k}^{\varkappa,n} & X_{i,i}^{\varkappa,n} \end{vmatrix}$

if $\gamma_k = \gamma_\varkappa$ *for all* $k \in \{1, \ldots, [n/2]\}$;

(2) $\left|X_{1,k}^{\varkappa,n}\right|, k = 1, \ldots, k_0 - 1, \qquad \left|X_{1,k_0}^{\varkappa_0,n}\right|^{\alpha_k}\left|X_{1,k}^{\varkappa,n}\right|, k = k_0 + 1, \ldots, \left[\frac{n}{2}\right]$

otherwise. Here k_0 is the minimal value of k for which $\gamma_k \neq \gamma_\varkappa$ and

$$\alpha_k = -\sum_{i=k_0}^{k} \frac{\gamma_{n-i+1} - \gamma_i}{\gamma_{n-k_0+1} - \gamma_{k_0}}.$$

Proof. Under normalization we impose the following restriction on the lifted invariants $\mathcal{I}_{ij}$, $j < i$:

$$\mathcal{I}_{ij} = 0 \qquad \text{if} \qquad j < i, \ (i, j) \neq (n - j' + 1, j'), \ j' = 1, \ldots, \left[\frac{n}{2}\right].$$

This means we do not only fix the values of the elements of the lifted invariant matrix $\mathcal{I}$, which are situated on the secondary diagonal under the main diagonal. The other significant elements of $\mathcal{I}$ are put equal to 0.

The decision on what to do with the singular lifted invariant $\mathcal{I}_0$ and the secondary diagonal lifted invariants $\mathcal{I}_{\varkappa k}$, $k = 1, \ldots, [n/2]$, is left for the later discussion, since it turns out that the necessity of imposing normalization conditions on them depends on the values of γ. As shown below, the final normalization in all the cases provides satisfying the conditions of proposition 1 from [4] and, therefore, is correct.

In view of the (triangular) structure of the matrices B and X, the formula $\mathcal{I} = BXB^{-1}$ determining the matrix part of the lifted invariants implies $BX = \mathcal{I}B$. This matrix equality is also significant for the matrix elements underlying the main diagonals of the left- and right-hand sides, i.e.,

$$e^{\gamma_i \varepsilon} x_{ij} + \sum_{i < i'} b_{ii'} x_{i'j} = \mathcal{I}_{ij} e^{\gamma_j \varepsilon} + \sum_{j' < j} \mathcal{I}_{ij'} b_{j'j}, \qquad j < i.$$

For convenience the latter system is divided under the chosen normalization conditions into four sets of subsystems

$$S_1^k: \quad e^{\gamma_\varkappa \varepsilon} x_{\varkappa j} + \sum_{i' > \varkappa} b_{\varkappa i'} x_{i'j} = 0, \qquad\qquad i = \varkappa, \quad j < k, \quad k = 2, \ldots, \left[\frac{n+1}{2}\right],$$

$$S_2^k: \quad e^{\gamma_\varkappa \varepsilon} x_{\varkappa k} + \sum_{i' > \varkappa} b_{\varkappa i'} x_{i'k} = \mathcal{I}_{\varkappa k} e^{\gamma_k \varepsilon}, \qquad i = \varkappa, \quad j = k, \quad k = 1, \ldots, \left[\frac{n}{2}\right],$$

$$S_3^k: \quad e^{\gamma_\varkappa \varepsilon} x_{\varkappa j} + \sum_{i' > \varkappa} b_{\varkappa i'} x_{i'j} = \mathcal{I}_{\varkappa k} b_{kj}, \qquad i = \varkappa, \quad k < j < \varkappa, \quad k = 1, \ldots, \left[\frac{n}{2}\right] - 1,$$

$$S_4^k: \quad e^{\gamma_k \varepsilon} x_{kj} + \sum_{i' > k} b_{ki'} x_{i'j} = 0, \qquad\qquad i = k, \quad j < k, \quad k = 2, \ldots, \left[\frac{n}{2}\right],$$

and solve them one after another. The subsystem S_2^1 consists of the single equation

$$\mathcal{I}_{n1} = x_{n1} e^{(\gamma_n - \gamma_1)\varepsilon}.$$

For any fixed $k \in \{2, \ldots, [n/2]\}$, the subsystem $S_1^k \cup S_2^k$ is a well-defined system of linear equations with respect to $b_{\varkappa i'}$, $i' > \varkappa$, and $\mathcal{I}_{\varkappa k}$. Analogously, the subsystem S_1^k for $k = \varkappa = [(n + 1)/2]$ in the case of an odd n is a well-defined system of linear equations with respect to $b_{k i'}$, $i' > k$. The solutions of the above subsystems are expressions of $x_{i'j}$, $i' \geqslant \varkappa$, $j < k$, and ε:

$$\mathcal{I}_{\varkappa k} = (-1)^{k+1} \frac{\left|X_{1,k}^{\varkappa,n}\right|}{\left|X_{1,k-1}^{\varkappa+1,n}\right|} e^{(\gamma_\varkappa - \gamma_k)\varepsilon}, \qquad k = 2, \ldots, \left[\frac{n}{2}\right],$$

$$B_{\varkappa+1,n}^{\varkappa,\varkappa} = -e^{\gamma_\varkappa \varepsilon} X_{1,k-1}^{\varkappa,\varkappa} \left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1}, \qquad k = 2, \ldots, \left[\frac{n+1}{2}\right].$$

After substituting the expressions of $\mathcal{I}_{\varkappa k}$ and $b_{\varkappa i'}$, $i' > \varkappa$, via ε and x's into S_3^k, we trivially resolve S_3^k with respect to b_{kj} as an uncoupled system of linear equations:

$$b_{1j} = \mathrm{e}^{\gamma_1 \varepsilon} \frac{x_{nj}}{x_{n1}}, \qquad 1 < j < n,$$

$$b_{kj} = (-1)^{k+1} \mathrm{e}^{\gamma_k \varepsilon} \frac{|X_{1,k-1}^{\varkappa+1,n}|}{|X_{1,k}^{\varkappa,n}|} \left(x_{\varkappa j} - X_{1,k-1}^{\varkappa,\varkappa} \left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{j,j}^{\varkappa+1,n}\right) = \frac{\mathrm{e}^{\gamma_k \varepsilon}}{|X_{1,k}^{\varkappa,n}|} \begin{vmatrix} X_{1,k-1}^{\varkappa,\varkappa} & x_{\varkappa j} \\ X_{1,k-1}^{\varkappa+1,n} & X_{j,j}^{\varkappa+1,n} \end{vmatrix},$$

$$k < j < \varkappa, \qquad k = 2, \ldots, \left[\frac{n}{2}\right] - 1.$$

Performing the subsequent substitution of the calculated expressions for b_{kj} to S_4^k, for any fixed appropriate k we obtain a well-defined system of linear equations, e.g., with respect to $b_{ki'}$, $i' > \varkappa$. Its solution is expressed via x's, $b_{k\varkappa}$ and ε:

$$B_{\varkappa+1,n}^{k,k} = -\left(\mathrm{e}^{\gamma_k \varepsilon} X_{1,k-1}^{k,k} + \sum_{k<j\leqslant\varkappa} b_{kj} X_{1,k-1}^{j,j}\right)\left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1}$$

$$= -b_{k\varkappa} X_{1,k-1}^{\varkappa,\varkappa} \left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} - \frac{\mathrm{e}^{\gamma_k \varepsilon}}{|X_{1,k}^{\varkappa,n}|} \sum_{k\leqslant j<\varkappa} \begin{vmatrix} X_{1,k-1}^{\varkappa,\varkappa} & x_{\varkappa j} \\ X_{1,k-1}^{\varkappa+1,n} & X_{j,j}^{\varkappa+1,n} \end{vmatrix} X_{1,k-1}^{j,j} \left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1},$$

$$k = 2, \ldots, \left[\frac{n}{2}\right].$$

The expression of the lifted invariant $\mathcal{I}_0$ is rewritten, taking into account the already imposed normalization constraints (note that $\varkappa = [(n+1)/2] + 1$ if $k = [n/2]$):

$$\mathcal{I}_0 = x_0 + \sum_l \gamma_l \widehat{b}_{ll} \sum_{l<i} b_{li} x_{il} + \sum_{k=2}^{[\frac{n+1}{2}]} \sum_{j<k} \gamma_k \widehat{b}_{jk} \sum_{i\geqslant k} b_{ki} x_{ij} + \sum_{k=1}^{[\frac{n}{2}]} \left(\sum_{j<k} + \sum_{k\leqslant j<\varkappa}\right) \gamma_\varkappa \widehat{b}_{j\varkappa} \sum_{i\geqslant\varkappa} b_{\varkappa i} x_{ij}$$

$$= x_0 + \sum_l \gamma_l \widehat{b}_{ll} \sum_{l<i} b_{li} x_{il} + \sum_{k=1}^{[\frac{n}{2}]} \gamma_\varkappa \mathcal{I}_{\varkappa k} \sum_{k\leqslant j<\varkappa} b_{kj} \widehat{b}_{j\varkappa}$$

$$= x_0 + \sum_{k=1}^{[\frac{n}{2}]} \gamma_k \widehat{b}_{kk} \left(\sum_{k<i\leqslant\varkappa} + \sum_{i>\varkappa}\right) b_{ki} x_{ik} + \sum_{k=1}^{[\frac{n+1}{2}]} \gamma_\varkappa \widehat{b}_{\varkappa\varkappa} \sum_{i>\varkappa} b_{\varkappa i} x_{i\varkappa} - \sum_{k=1}^{[\frac{n}{2}]} \gamma_\varkappa \widehat{b}_{\varkappa\varkappa} \mathcal{I}_{\varkappa k} b_{k\varkappa}.$$

Then the found expressions for b's and $I_{\varkappa k}$ are substituted into the derived expression of $\mathcal{I}_0$:

$$\mathcal{I}_0 = x_0 + \gamma_1 \mathrm{e}^{-\gamma_1 \varepsilon} \sum_{1<i\leqslant n} b_{1i} x_{i1} + \sum_{k=2}^{[\frac{n}{2}]} \gamma_k \, \mathrm{e}^{-\gamma_k \varepsilon} \sum_{k<i\leqslant\varkappa} b_{ki} \left(x_{ik} - X_{1,k-1}^{i,i} \left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{k,k}^{\varkappa+1,n}\right)$$

$$- \sum_{k=2}^{[\frac{n}{2}]} \gamma_k X_{1,k-1}^{k,k} \left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{k,k}^{\varkappa+1,n} + \sum_{k=1}^{[\frac{n+1}{2}]} \gamma_\varkappa \widehat{b}_{\varkappa\varkappa} \sum_{i>\varkappa} b_{\varkappa i} x_{i\varkappa} - \sum_{k=1}^{[\frac{n}{2}]} \gamma_\varkappa \widehat{b}_{\varkappa\varkappa} \mathcal{I}_{\varkappa k} b_{k\varkappa}$$

$$= x_0 + (\gamma_1 - \gamma_n) \, \mathrm{e}^{-\gamma_1 \varepsilon} b_{1n} x_{n1} + \sum_{k=2}^{[\frac{n}{2}]} (\gamma_k - \gamma_\varkappa) \, \mathrm{e}^{-\gamma_k \varepsilon} b_{k\varkappa} (-1)^{k+1} \frac{|X_{1,k}^{\varkappa,n}|}{|X_{1,k-1}^{\varkappa+1,n}|}$$

$$- \sum_{k=2}^{[\frac{n}{2}]} \gamma_k X_{1,k-1}^{k,k} \left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{k,k}^{\varkappa+1,n} - \sum_{k=2}^{[\frac{n+1}{2}]} \gamma_\varkappa X_{1,k-1}^{\varkappa,\varkappa} \left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{\varkappa,\varkappa}^{\varkappa+1,n}$$

$$+ \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^{k+1} \gamma_k}{|X_{1,k}^{\varkappa,n}|} \sum_{k<i<\varkappa} \begin{vmatrix} X_{1,k}^{i,i} & 0 \\ X_{1,k}^{\varkappa,n} & X_{i,i}^{\varkappa,n} \end{vmatrix} + \sum_{k=2}^{[\frac{n}{2}]} \frac{(-1)^{k+1} \gamma_k}{|X_{1,k-1}^{\varkappa+1,n}|} \sum_{k<i<\varkappa} \begin{vmatrix} X_{1,k-1}^{i,i} & 0 \\ X_{1,k-1}^{\varkappa+1,n} & X_{i,i}^{\varkappa+1,n} \end{vmatrix}.$$

If $\gamma_k = \gamma_\varkappa$ for all $k \in \{1, \ldots, [n/2]\}$ then $\mathcal{I}_{\varkappa k}$, $k = 1, \ldots, [n/2]$, and $\mathcal{I}_0$ do not depend on the parameters b and ε, i.e., they are invariants. For a basis to be simpler, $\widehat{\mathcal{I}}_0 = \mathcal{I}_0$ is taken,

as well as $\hat{\mathcal{I}}_1 = \mathcal{I}_{n1}$ and the combinations $\hat{\mathcal{I}}_k = (-1)^{k+1}\mathcal{I}_{\varkappa k}\hat{\mathcal{I}}_{k-1}, k = 2, \ldots, [n/2]$, resulting in the first tuple of invariants from the statement of the theorem. Note that only under the supposition on γ, the above formula for $\mathcal{I}_0$ is transformed into

$$\mathcal{I}_0 = x_0 + \left(\sum_{k=1}^{[\frac{n}{2}]-1} + \sum_{k=[\frac{n}{2}]}^{[\frac{n}{2}]}\right) \frac{(-1)^{k+1}\gamma_k}{\left|X_{1,k}^{\varkappa,n}\right|} \sum_{k<i<\varkappa} \begin{vmatrix} X_{1,k}^{i,i} & 0 \\ X_{1,k}^{\varkappa,n} & X_{i,i}^{\varkappa,n} \end{vmatrix}$$

$$+ \sum_{k=2}^{[\frac{n}{2}]} \frac{(-1)^{k+1}\gamma_k}{\left|X_{1,k-1}^{\varkappa+1,n}\right|} \sum_{k\leqslant i\leqslant\varkappa} \begin{vmatrix} X_{1,k-1}^{i,i} & 0 \\ X_{1,k-1}^{\varkappa+1,n} & X_{i,i}^{\varkappa+1,n} \end{vmatrix} - \sum_{k=[\frac{n}{2}]}^{[\frac{n+1}{2}]} \gamma_\varkappa X_{1,k-1}^{\varkappa,\varkappa}\left(X_{1,k-1}^{\varkappa+1,n}\right)^{-1} X_{\varkappa,\varkappa}^{\varkappa+1,n}.$$

This results in the expression for $\mathcal{I}_0$ from the statement, after shifting the index k in the third sum by -1 and a further permutation and recombination of terms.

Otherwise, if there exists $k_0 \in \{1, \ldots, [n/2]\}$ such that $\gamma_{k_0} \neq \gamma_{\varkappa_0}$ then $\mathcal{I}_0$ necessarily depends on the parameter $b_{k_0\varkappa_0}$, which is in the expressions of $\mathcal{I}_{\varkappa k}, k = 1, \ldots, [n/2]$, under the already established normalization conditions. Hence an additional normalization condition constraining $\mathcal{I}_0$ should be used, e.g., $\mathcal{I}_0 = 0$. It yields an expression for $b_{k_0\varkappa_0}$ via x's, other $b_{k\varkappa}$'s and ε. The exact form of the latter expression is inessential. Suppose that k_0 is the minimal k for which $\gamma_k \neq \gamma_\varkappa$. $\hat{\mathcal{I}}_1 = \mathcal{I}_{n1}$ and the combinations $\hat{\mathcal{I}}_k = (-1)^{k+1}\mathcal{I}_{\varkappa k}\hat{\mathcal{I}}_{k-1}, k = 2, \ldots, [n/2]$, are taken. Since $\hat{\mathcal{I}}_{k_0}$ explicitly depends on ε, we impose one more normalization condition $\hat{\mathcal{I}}_{k_0} = 1$ and, using it, exclude the parameter ε from the other $\hat{\mathcal{I}}$'s. As a result, we construct the second tuple of invariants from the statement of the theorem.

Under the normalization we express the non-normalized lifted invariants via x's and compute a part of the parameters b's of the co-adjoint action via x's and the other b's. The expressions in the obtained tuples of invariants are functionally independent. No equations involving only x's are obtained. In view of proposition 1 of [4], this implies that the choice of normalization constraints, which depends on values of γ, is correct. That is why the number of the found functionally independent invariants is maximal, i.e., they form bases of $\mathrm{Inv}(\mathrm{Ad}^*_{T_\gamma(n)})$. $\qquad\square$

Corollary 1. $\left|X_{1,k}^{\varkappa,n}\right|, k = 1, \ldots, [n/2]$, are functionally independent relative invariants of $\mathrm{Ad}^*_{T_\gamma(n)}$ for any admissible value of γ.

Let us recall [13][definition 3.30] that, given a group G acting on a set M, a function $F: M \to \mathbb{R}$ is called a *relative invariant* of the representation of G if $F(g \cdot x) = \mu(g, x)F(x)$ for all $g \in G$ and $x \in M$ and some multiplier $\mu: G \times M \to \mathbb{R}$ of this representation.

4. Invariants of $\mathsf{t}_\gamma(n)$

Theorem 2. A basis of $\mathrm{Inv}(\mathsf{t}_\gamma(n))$ is formed by the expressions

(1) $\left|\mathcal{E}_{\varkappa,n}^{1,k}\right|, k = 1, \ldots, \left[\frac{n}{2}\right],$ $\qquad f + \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^{k+1}}{\left|\mathcal{E}_{\varkappa,n}^{1,k}\right|}(\gamma_k - \gamma_{k+1}) \sum_{i=k+1}^{n-k} \begin{vmatrix} \mathcal{E}_{i,i}^{1,k} & \mathcal{E}_{\varkappa,n}^{1,k} \\ 0 & \mathcal{E}_{\varkappa,n}^{i,i} \end{vmatrix}$

if $\gamma_k = \gamma_\varkappa$ for all $k \in \{1, \ldots, [n/2]\}$; hereafter $\varkappa = n - k + 1, \mathcal{E}_{j_1,j_2}^{i_1,i_2}, i_1 \leqslant i_2, j_1 \leqslant j_2$, denotes the matrix $(e_{ij})_{j=j_1,\ldots,j_2}^{i=i_1,\ldots,i_2}$;

(2) $\left|\mathcal{E}_{\varkappa,n}^{1,k}\right|, k = 1, \ldots, k_0 - 1,$ $\qquad \left|\mathcal{E}_{\varkappa_0,n}^{1,k_0}\right|^{\alpha_k}\left|\mathcal{E}_{\varkappa,n}^{1,k}\right|, k = k_0 + 1, \ldots, \left[\frac{n}{2}\right]$

otherwise. Here k_0 is the minimal value of k for which $\gamma_k \neq \gamma_\varkappa$ and

$$\alpha_k = -\sum_{i=k_0}^{k} \frac{\gamma_{n-i+1} - \gamma_i}{\gamma_{n-k_0+1} - \gamma_{k_0}}.$$

Proof. Expanding the determinants in any elements of the basis from theorem 1, we obtain a rational expression for the x's. Each monomial in the numerator or the denominator contains coordinate functions such that corresponding basis elements commute. Therefore, the symmetrization procedure is trivial. Since $x_{ij} \sim e_{ji}$, $j < i$, it is necessary to transpose the matrices in the obtained expressions of invariants in order to improve the representation. $\square$

Corollary 2. *If $\gamma_k = \gamma_\varkappa$ for all $k \in \{1, \ldots, [n/2]\}$ then* $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$ *has a basis from Casimir operators. Otherwise, the algebra $\mathfrak{t}_\gamma(n)$ admits a rational basis of invariants if and only if $\alpha_k \in \mathbb{Q}$ for all $k \in \{k_0, \ldots, [n/2]\}$, and admits a polynomial basis of invariants if and only if, additionally, $\alpha_k \geqslant 0$ for all $k \in \{k_0, \ldots, [n/2]\}$. Here k_0 is the minimal value of k for which $\gamma_k \neq \gamma_\varkappa$.*

Note 2. It follows from theorem 2 that the maximal number $N_{\mathfrak{t}_\gamma(n)}$ of functionally independent invariants of the algebra $\mathfrak{t}_\gamma(n)$ is equal to $[n/2] + 1$ if $\gamma_k = \gamma_\varkappa$ for all $k \in \{1, \ldots, [n/2]\}$ and to $[n/2] - 1$ otherwise. The condition on the extension of $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$ can be reformulated in terms of commutators in the following way. The nil-independent basis element f commutes with the 'nilpotent' basis elements $e_{k\varkappa}$, $k = 1, \ldots, [n/2]$, lying on the significant part of the secondary diagonal of the basis 'matrix' $\mathcal{E}$, i.e., $[f, e_{k\varkappa}] = 0$, $k = 1, \ldots, [n/2]$.

Note 3. The significant elements of the secondary diagonal of the lifted invariant matrix play a singular role under the normalization procedure in all investigated algebras with nil-radicals isomorphic to $\mathfrak{t}_0(n)$: $\mathfrak{t}_0(n)$ itself and $\mathfrak{st}(n)$ [4] as well as $\mathfrak{t}_\gamma(n)$, which is studied in this paper. (More precisely, in [4] the normalization procedure was realized for $\mathfrak{t}(n)$ and then the results on the invariants were extended to $\mathfrak{st}(n)$.) The reasons for such a singularity were not evident from the consideration of [4]. Only note 2 gives an explanation for this and justifies the naturalness of the chosen normalization conditions.

5. Conclusion and discussion

Using the technique developed in [4] for triangular algebras in the framework of our original pure algebraic approach [2, 3], in this paper we investigated the invariants of solvable Lie algebras with nil-radicals isomorphic to $\mathfrak{t}_0(n)$ and one nil-independent 'diagonal' element. The algorithm has two main steps. They are the construction of explicit formulae for a fundamental lifted invariant of the co-adjoint representation of the corresponding connected Lie group and the normalization procedure for excluding parameters from lifted invariants. Realization of both steps for the algebras under consideration is more difficult than for the universal triangular algebras $\mathfrak{t}_0(n)$ and $\mathfrak{t}(n)$. Thus, a fundamental lifted invariant has a more complex representation. One of its components does not admit a good interpretation as an element of the matrix of the significant part of which is formed by the other components. The choice of normalization conditions essentially depends on the algebra parameters that leads to the furcation of the calculations and final results.

There are two principally different cases on the number of normalization conditions and, therefore, on the cardinality of the fundamental invariants. If $\gamma_k = \gamma_\varkappa$ for all $k \in \{1, \ldots, [n/2]\}$ (the singular case), the algebra $\mathfrak{t}_\gamma(n)$ has $[n/2] + 1$ functionally independent invariants. The basis of $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$, constructed in theorem 2 for this case, consists of polynomial invariants

forming a basis of $\text{Inv}(\mathfrak{t}_0(n))$ and one more nominally rational invariant which includes the chosen nil-independent element f, and can be replaced by a more complicated polynomial invariant. Otherwise (the regular case), the maximal number $N_{\mathfrak{t}_\gamma(n)}$ of functionally independent invariants of the algebra $\mathfrak{t}_\gamma(n)$ is equal to $[n/2] - 1$. In this case a basis of $\text{Inv}(\mathfrak{t}_\gamma(n))$ can be presented via combinations of powers of the basis invariants of $\text{Inv}(\mathfrak{t}_0(n))$. The basis is polynomial or rational only under special restrictions on the algebra parameters. The conjecture of [20] on the number and form of elements in the bases is corroborated. Only in the regular case should the basis be written more precisely.

In spite of the above difficulties, the calculations are quite handy due to the use of the optimized technique. This technique includes the choice of special coordinates in the inner automorphism group, the matrix representation of most of the lifted invariants and the natural normalization constraints associated with the algebra structure. The cardinality of the invariant basis is determined in the process of finding the invariants. Moreover, we only partially constrain the lifted invariants in the beginning of the normalization procedure. Both the total number of necessary constraints and the additional constraints are specified before the completion of the normalization. As a result of the optimization, eliminating of the group parameters in the singular case is reduced to a linear system of (algebraic) equations. After solving a similar linear system in the regular case, we eliminate most of the group parameters and obtain nonlinear algebraic equations for the elimination of only one parameter, these equations are trivial.

The present investigation can be directly extended to similar solvable Lie algebras with more nil-independent diagonal elements. All such algebras are embedded in $\mathfrak{st}(n)$ as ideals. The technique should be modified slightly. An entirely different matter is the investigation of the other solvable Lie algebras with nil-radicals isomorphic to $\mathfrak{t}_0(n)$. It is not yet known whether we will be able to use the partial matrix representation of the lifted invariants, as well as other tricks lifted from the technique explained herein, as applied to this problem.

Acknowledgments

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